MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #5: Matchings in bipartite graphs

Due Thursday, October 19th, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. Imagine that the McGill Extended Squad of Silliness (MESS) posts a new regulation regarding the student clubs at McGill. Specifically, the new rules are:

1) No student can be a member of more than 50 different clubs,
2) Every club must have its president, and the president must be a member of the club,
3) No student can serve as the president for more than 5 different clubs at a time.

However, implementing the rules turn the student club situation at McGill into a complete mess (thank you, MESS!), and it is now not possible to follow the rules and assign presidents for all the clubs. In order to fix this, MESS intends to introduce an additional rule stating:

4) Every club must have at least K members (all the clubs with fewer members will be closed).

This is the moment when MESS wants to hire you for a consultation: what is the minimum value of K so that it is always guaranteed that there will exists an assignment of the presidents to the clubs fulfilling the requirements (1)-(4)? MESS offers to pay you 3 points for MATH 350 if you can give them the best value of K, and also prove that with $K-1$ it can happen that there is no such an assignment. *(3 points)*

Solution: We claim that $K = 10$. Firstly, if $K = 9$, then it could happen that for example only 9 McGill students will get involved in the McGill clubbing this year, and all of a sudden all of them subscribe to 50 clubs. Then clearly, we cannot use those 9 people as presidents for more than 45 clubs. BTW, you might think that having only 9 clubbing students is a bit unrealistic scenario, but we can extend the 9-people example as follows: Add as many new students as we like, and as many new clubs as we like, and as long as we make sure that the new students are not registered in the original 50 clubs, it is still impossible to assign the presidents to the first 50 clubs.

All right, so the remaining task is to show that $K = 10$ is enough. Let S be the set of all students, let C be the set of all clubs, and let T be the set of size $5 \times |S|$ obtained by taking 5 disjoint copies of S. We define a bipartite graph H with parts C and T by simply connecting a student $s \in T$ to the clubs (s)he is a member of. Clearly, $\deg_H(s) \leq 50$ for every $s \in T$. On the other hand, every club $c \in C$ has at least $K = 10$ members, so there are at least 5×10 neighbors of c in T. Moreover, feasible assignments of presidents to the clubs C are in one-to-one correspondence with C-covering matchings in H so all we need to do is to find a C-covering matching in H. Existence of such a matching directly follows from the following lemma:

Lemma. *Let G be a bipartite graph with parts A and B and suppose that* $\deg_G(a) \leq d \leq \deg_G(b)$ *for every $a \in A$ and $b \in B$. Then G has a B-covering matching.*

Proof of Lemma. We verify Hall's condition for G and the part B. Let $X \subseteq B$ be chosen arbitrary. Our aim is to prove that $|N_G(X)| \geq |X|$.

Let us count the number of edges in G that go between $N_G(X)$ and X in two different ways: on the one hand, there are at least $d|X|$ edges leaving X. On the other hand, there can be at most $d|N_G(X)|$ edges leaving $N_G(X)$, so, in particular, at most $d|N_G(X)|$ goes from $N_G(X)$ to X. Therefore,

$$d|X| \leq \#\text{ of edges between } X \text{ and } N_G(X) \leq d|N_G(X)|\ ,$$

and hence also $|X| \leq |N_G(X)|$.

□

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2. An $n \times n$ *Latin square* is a table with n rows and n columns, where each cell contains one number between 1 and n in such a way that in each row every number appears exactly once, and also in every column each number appears exactly once (similarly as in Sudoku). See examples of a 3×3 and a 4×4 Latin squares.

1	2	3
2	3	1
3	1	2

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

Analogously, for $m \leq n$, an $m \times n$ *Latin rectangle* is a table with m rows and n columns where each cell contains one number between 1 and n in such a way that in each row every number appears exactly once, and in every column each number appears at most once.
Prove that for any $m \times n$ Latin rectangle there exists an $n \times n$ Latin square so that the first m lines of the Latin square are equal to the lines of the rectangle. *(3 points)*

Solution: The proof goes by induction on $\ell := (n - m)$. If $\ell = 0$, there is nothing to prove. Let us move to the case when $\ell \geq 1$, i.e., the case $m < n$. Our aim will be to add a new row to a given $m \times n$ Latin rectangle R so that the extended $(m+1) \times n$ rectangle R' is still going to be Latin, and then use the induction to extend R' into a Latin square.
Let H be an auxiliary bipartite graph with the parts $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, \ldots, n\}$. We connect $i \in A$ with $j \in B$ if and only if the number j does not appear in the i-th column of R. Since R is Latin, the degree $\deg_H(i) = n - m = \ell$ for every $i \in A$. On the other hand, every $j \in B$ appear exactly once in each row of R, hence $\deg_H(j) = n - m = \ell$ for every $j \in B$. So H is an ℓ-regular bipartite graph, and by Corollary 8.3 from the notes it has a perfect matching $M = \{ij_i : i \in A\}$. The sought new row is then simply obtained by writing the value j_i to the i-th column.

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3. For a graph $G = (V, E)$ and $S \subseteq V$, recall that $N(S) := \bigcup_{u \in S} N(u) = \{v \in V \mid \exists u \in S \wedge uv \in E\}$.

a) Let $G = (V, E)$ be a bipartite graph. Prove that G has a perfect matching if and only if $|N(S)| \geq |S|$ for every $S \subseteq V$. *(3 points)*

Solution: Let A and B be the parts of a bipartition of G. Firstly, if G satisfies the condition $|N(S)| \geq |S|$ for every $S \subseteq V$, then it satisfies Hall's condition for A. Therefore, by Hall's theorem G contains an A-covering matching M_A, which in particular means that $|A| \leq |B|$. Also, for the same reasons, G contains a B-covering matching M_B, so $|A| = |B|$ hence both M_A and M_B are perfect matchings in G.

Now suppose that G has a perfect matching M and we are given $S \subseteq V$. We claim that $|N(S)| \geq |S|$. Indeed, let $Z := \{u \in V : \exists v \in S \,\&\, uv \in M\}$ i.e., the set of other ends for vertices from S in M. The claim readily follows from $|Z| = |S|$ and $Z \subseteq N(S)$.

b) Construct a connected non-bipartite graph $G = (V, E)$ with $|V|$ even that has no perfect matching but yet it satisfies that $|N(S)| \geq |S|$ for every $S \subseteq V$. *(1 point)*

Solution: See the 10-vertex graph G in Figure 1. We need to check two things: First of all, G satisfies $|N(S)| \geq |S|$ for all $S \subseteq V(G)$. Indeed, if $v \in S$, then $|N(S)| \geq 9$ and $|N(S)| \neq 10$ only when $S = \{v\}$. Suppose now $v \notin S$, and partition $V \setminus \{v\} = V_1 \cup V_2 \cup V_3$, where V_i correspond to

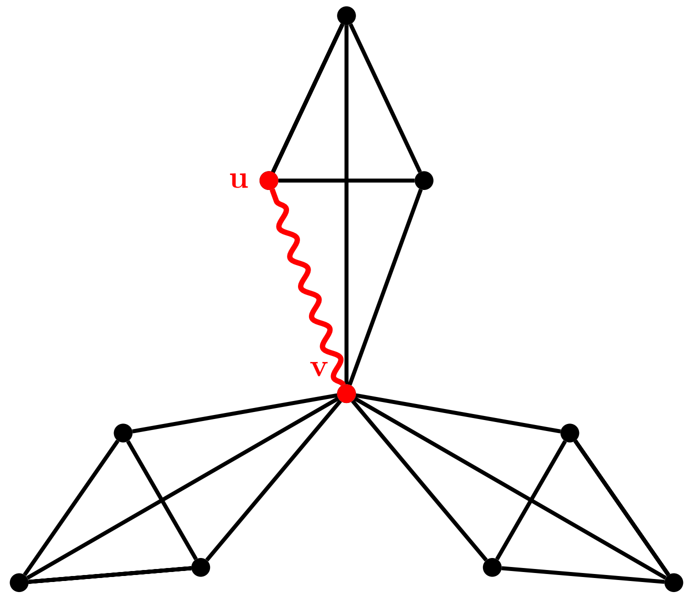

Figure 1: A graph for Problem 3b).

the vertices of the i-th connected component of $G - v$, and set $S_i := V_i \cap S$. If $|S_i| = 1$, then $N(S)$ contains two vertices from V_i, and as long as $|S_i| \geq 2$, then $N(S) \cap V_i = V_i$. Therefore,

$$|N(S)| \geq |N(S_1) \cap V_1| + |N(S_2) \cap V_2| + |N(S_3) \cap V_3| \geq |S_1| + |S_2| + |S_3| = |S| \ .$$

The last thing we need to observe is that the depicted graph has no perfect matching. Indeed, one can see that $G - v$ has three connected components of odd size which contradicts the necessary condition in Tutte's theorem, or, alternatively, if you assume G has a perfect matching M, then WLOG v is matched to u by M. But then the edges of M should somehow match the vertices in the other two triangles, which is indeed not possible.