MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #6: Matchings in arbitrary graphs, Tutte's theorem

Due Thursday, October 26th, 8:30AM *Write your answers clearly. Justify all your answers.*

---

1. Let T be a tree. Prove that T has a perfect matching if and only if for every vertex $v \in V(T)$ the subgraph $T - v$ contains exactly one connected component with odd number of vertices. *(2 points)*

Solution: Clearly, if T has a perfect matching then it satisfies Tutte's condition, and, in particular, for every $v \in V$, the subgraph $T - v$ has exactly one connected component with odd number of vertices.
Now we show that if $odd_T(V - v) = 1$ for all $v \in V$ then T has a perfect matching. Firstly, if v is a leaf then $T - v$ has just one connected component, and hence T has even number of vertices. We now proceed by proving the statement by induction on n. If $n = 2$, then T is a single edge which also forms a perfect matching in T. Let us now assume $n \geq 4$. Let u be a leaf of T and w its unique neighbor. Consider $T' := T - u - w$; we show that T' has a perfect matching M' which together with the edge $\{u, w\}$ forms a perfect matching in T.
Suppose for contradiction T' has no perfect matching. By the induction hypothesis, there is a vertex $v \in V(T') = V \setminus \{u, v\}$ such that $T' - v$ has $\ell \geq 3$ connected components $C_1, C_2, \ldots, C_\ell$ of odd size. However, since $T - v$ has only one odd component, the vertex w must have a neighbor in at least two of the components C_i and C_j (where $i \neq j$). Therefore, there are at least two different paths from v to w (one path passes through C_i, the other one through C_j); a contradiction.

---

2. Let G be a 3-regular simple graph with no cut-edge, and let $e \in E(G)$ be an edge of G.

a) Prove that G has a perfect matching M_1 such that $e \in M_1$. *(2 points)*

Solution: Let u and w be the two endpoints of e, and let $H := G - u - w$. It is enough to show that H has a perfect matching M', since $M' + e$ will be a perfect matching of G that contains e.

Let $V := V(G)$ and $W := V(H) = V \setminus \{u, w\}$. Suppose for contradiction H does not have a perfect matching. By Tutte's theorem, there exists $S_0 \subseteq W$ such that $odd_H(\overline{S_0}) > |S_0|$, where $\overline{S_0} = W \setminus S_0$. First, we observe that the parity of $odd_H(\overline{S_0})$ and $|S_0|$ is the same. Indeed, recall that $|V|$ is even and that

$$|V| - 2 = |W| = \sum_{\substack{C \text{ even component} \\ \text{of } H[\overline{S_0}]}} |C| \; + \sum_{\substack{C \text{ odd component} \\ \text{of } H[\overline{S_0}]}} |C| \; + \; |S_0|.$$

Therefore, $odd_H(\overline{S_0}) \geq |S_0| + 2$, and for $S := S_0 \cup \{u, w\}$ we have

$$odd_G(V \setminus S) = odd_H(\overline{S_0}) \geq |S_0| + 2 = |S|.$$

Now we look closer to the situation in G and the set of vertices S. The number of edges between S and $V \setminus S$ is at most $3(|S| - 2) + 4 = 3|S| - 2$ because u is adjacent to at most two vertices in $V \setminus S$ and the same holds also for v. On the other hand, there are at least $|S|$ odd components in $G[V \setminus S]$. As in the proof of Petersen's theorem in the lecture, each such odd connected component must receive at least 3 edges from the vertices in S (only one edge would mean a cut-edge in G, only two edges violates the parity constraint). So the number of edges between S and $V \setminus S$ must be at least $3|S|$; a contradiction.

b) Prove that G has a perfect matching M_2 such that $e \notin M_2$. *(2 points)*

Solution: This immediately follows from (a). Let v be one of the endpoints of e and let f be one of the other two edges incident to v (chosen aribtrarirly). A perfect matching M containing f guaranteed by (a) clearly cannot contain e.

3.

a) Let $k \geq 3$ and $G = (V, E)$ a k-regular connected graph with even number of vertices. Suppose G has the property that for every set of edges $F \subseteq E$ of size $k-2$, the subgraph $(V, E \setminus F)$ is still connected (graphs with this property are called $(k-1)$-*edge-connected*). Prove that G has a perfect matching.

(2 points)

Solution: Fix the value of k, and a k-regular $(k-1)$-edge-connected graph $G = (V, E)$. Suppose G has no perfect matching. Then by Tutte's theorem, there is $S \subseteq V$ with $|S| = \ell$ and the odd connected components of $G - S$ are $C_1, C_2, \ldots, C_m$ with $m > \ell$.

Claim. *Every odd connected component C_i sends at least k edges to S.*

Proof. By the connectivity assumption on G, the only situation when this claim would be false is when C_i sends exactly $k-1$ edges out. Let H_i be the induced subgraph of G defined by C_i. By definition,

$$\sum_{v \in C_i} \deg_{H_i}(v) = k \cdot |C_i| - (k-1) = k \cdot (|C_i| - 1) + 1 \; .$$

Since $|C_i| - 1$ is even, the above sum must be odd, which contradicts the hand-shaking lemma. □

Let's count the edges of G between S and $\bigcup_{i \in \{1,2,\ldots,m\}} C_i$ now. On the one hand, that number is at most $\ell \cdot k$. On the other hand, the claim above yields the number of such edges is at least $m \cdot k$, and hence $m \leq \ell$; a contradiction.

b) For every $k \geq 3$, construct a k-regular graph $Z_k = (V, E)$ with even number of vertices which has the property that for every subset of edges $F \subseteq E$ of size $k-3$, the subgraph $(V, E \setminus F)$ is still connected (i.e., Z_k is $(k-2)$-*edge-connected*), but yet Z_k has no perfect matching. *(2 points)*

Solution: <u>**This was perhaps a bit harder exercise than I originally thought; I apologize.**</u>

Firstly, let us show that there exists a $(k-2)$-edge-connected graph where $(k+1)$ vertices have degree k, and $(k-2)$ vertices have degree $(k-1)$.

Lemma. *Fix an integer $k \geq 3$. There exist a $(2k-1)$-vertex $(k-2)$-edge-connected simple graph $H_k = (V_k, E_k)$ with $V_k = \{y_1, y_2, \ldots, y_{k+1}, z_1, z_2, \ldots, z_{k-2}\}$, where all the vertices y_i, for $1 \leq i \leq k+1$, have degree exactly k, and all the vertices z_j, for $1 \leq j \leq k-2$, have degree exactly $(k-1)$.*

Proof. Start with a k-vertex complete graph on the vertices $\{y_1, y_2, \ldots, y_k\}$, plus a $(k-2)$-vertex complete graph on the vertices $\{z_1, z_2, \ldots, z_{k-2}\}$. Next, place an edge from the vertex y_{k+1} to the vertices y_{k-1} and y_k, then $(k-2)$ edges of the form $y_j z_j$, where $1 \leq j \leq k-2$, and also $(k-2)$ edges of the form $y_{k+1} z_j$, again where $1 \leq j \leq k-2$. Observe that every vertex y_i, where $1 \leq i \leq k+1$, has degree exactly k, and every vertex z_j, where $1 \leq j \leq k-2$, has degree exactly $(k-1)$.

It remains to show that H_k is $(k-2)$-edge-connected. In other words, we want to show that for any two vertices $u, w \in V_k$ and any $(k-3)$ edges $F \subseteq E_k$, there will still be some path from u to w in the subgraph $H_k - F = (V_k, E_k \setminus F)$. This will definitely be true if we find $(k-2)$ edge-disjoint paths between u and w in the graph H_k, simply because removing F can break at most $(k-3)$ of them.

Claim. *For every $u, w \in V_k$, there exist $(k-2)$ edge-disjoint paths from u to w in H_k.*

Unfortunately, we need to go through different cases here…

Case 1) $u, w \in \{y_1, y_2, \ldots, y_k\}$. Indeed; actually, we can easily find even $(k-1)$ edge-disjoint paths. Without loss of generality, $u = y_1$ and $w = y_2$. Then one path is simply the edge y_1, y_2, and then there $(k-2)$ paths of the form $u = y_1, y_i, y_2 = w$, where $3 \leq i \leq k$.

Case 2) $u \in \{y_1, y_2, \ldots, y_{k-2}\}$ *and* $w = y_{k+1}$. By symmetry, we may assume $u = y_1$. We actually describe k edge-disjoint paths this time: there are three paths $P_1 = y_1, z_1, y_{k+1}$, $P_2 = y_1, y_{k-1}, y_{k+1}$, $P_3 = y_1, y_k, y_{k+1}$, and, for every i such that $2 \leq i \leq k-2$, there is a path $P_{i+2} = y_1, y_i, z_i, y_{k+1}$.

Case 3) $u \in \{y_{k-1}, y_k\}$ *and* $w = y_{k+1}$. Again by symmetry, we may assume $u = y_k$. There are $(k-2)$ edge-disjoint paths of the form $y_k y_i z_i y_{k+1}$ ($\ldots$ plus also an edge $y_k, y_{k+1} \ldots$).

Case 4) $u \in \{y_1, y_2, \ldots, y_{k-2}\}$ *and* $w \in \{z_1, \ldots, z_{k-2}\}$. Without loss of generality, $u = y_1$. If $w = z_1$, then there is an edge y_1, z_1 and $(k-3)$ edge-disjoint paths of the form y_1, y_i, z_i, z_1, where $2 \leq i \leq k-2$. If $w \neq z_1$, then by symmetry we may assume $w = z_2$. This time, we have two paths y_1, z_1, z_2 and y_1, y_2, z_2, plus $(k-4)$ paths of the form y_1, y_i, z_i, z_2, where $3 \leq i \leq k-2$.

Case 5) $u \in \{y_{k-1}, y_k\}$ *and* $w \in \{z_1, \ldots, z_{k-2}\}$. Without loss of generality, $u = y_k$ and $w = z_1$. The sought paths are $P_1 = y_k, y_1, z_1$ and $P_i = y_k, y_i, z_i, z_1$, where $2 \leq i \leq k-2$.

Case 6) $u = y_{k+1}$ *and* $w \in \{z_1, \ldots, z_{k-2}\}$. By symmetry, we assume $w = z_1$. This time, we take $P_1 = y_{k+1}, z_1$ and for $2 \leq i \leq k-2$, we take $P_i = y_{k+1}, z_i, z_1$.

Case 7) $u, w \in \{z_1, \ldots, z_{k-2}\}$. Finally, the last case! Without loss of generality, $u = z_1$ and $v = z_2$. This time, we take $P_1 = z_1, z_2$, $P_2 = z_1, y_{k+1}, z_2$, and for $3 \leq i \leq k-2$, we take $P_i = z_1, z_i, z_2$.

This finishes the proof of the claim as well as the proof of the lemma; Yay! □

Now we claim that the following graph Z_k is both $(k-2)$-edge-connected and yet has no perfect matching: take k vertex-disjoint copies $C^1, C^2, \ldots, C^k$ of the graph H_k, add $(k-2)$ new vertices $v_1, v_2, \ldots, v_{k-2}$, and for every $i \in \{1, \ldots, k-2\}$ and $j \in \{1, \ldots, k\}$ connect v_i to z_i^j, where z_i^j is the i-th vertex of degree $(k-1)$ in the copy C^j.

Clearly, Z_k is k-regular and has no perfect matching. The latter one is because

$$odd_G(V(Z_k) - S) = k = |S| + 2 \quad \text{for } S = \{v_1, \ldots, v_{k-2}\}.$$

It remains to show that Z_k stays connected even after removing arbitrary $(k-3)$ edges $F \subseteq E(Z_k)$. Suppose there is $F \subseteq E(Z_k)$ of size $(k-3)$ such that $G - F$ is not connected. Since H_k is $(k-2)$-edge-connected, we may assume that every $f \in F$ is incident to some vertex v_i. This reduces the problem of $(k-2)$-edge-connectivity of the graph Z_k to the following simple lemma:

Lemma. *For every $k \geq 3$, the complete bipartite graph $K_{k,k-2}$ is $(k-2)$-edge-connected.*

Proof. Let $A = \{a_1, \ldots, a_k\}$ be the part of size k and $B = \{b_1, \ldots, b_{k-2}\}$ be the part of size $(k-2)$. We will show that between any two vertices $u, w \in A \cup B$, there are $(k-2)$-edge-disjoint paths. Again, we go over three cases here:

Case 1) $u, w \in A$. The sought paths are $P_i = u, b_i, w$, where $1 \leq i \leq k-2$.

Case 2) $u \in A$ and $w \in B$. By symmetry, we may assume $u = a_1$. Let $w = b_j$ for some $j \in \{1, \ldots, k-2\}$. Apart from the edge a_1, b_j, there are also $(k-3)$ paths of the form a_1, b_i, a_i, b_j, where $1 \leq i \leq k-2$ and $i \neq j$.

Case 3) $u, w \in B$. This time, we find k edge-disjoint paths $P_i = u, a_i, w$, where $1 \leq i \leq k$.

This finishes the proof of the lemma, which in turn yields that Z_k is indeed $(k-2)$-edge-connected. □

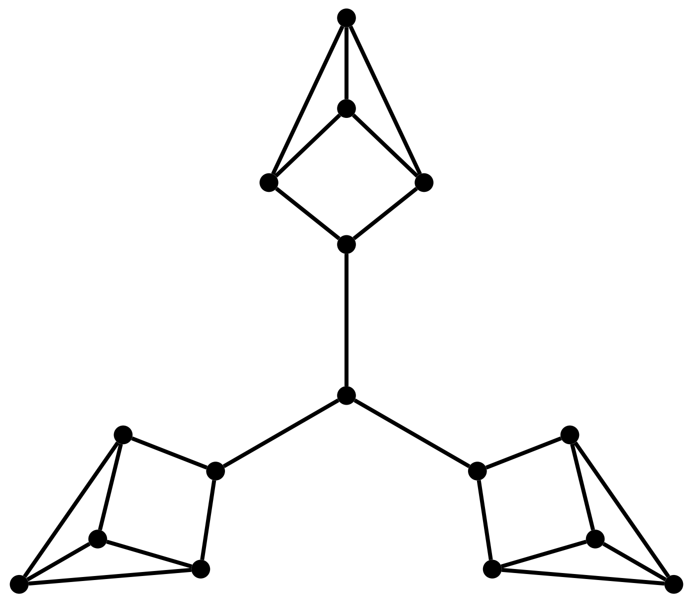

Figure 1: The graph Z_3 for Question 3b.