MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #3: Menger's theorem and network flows

Due Wednesday, November 2nd, 2016, 14:30

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1. Let $G = (V, E)$ be a simple graph and let $U \subseteq V$. We define $G \oplus_U \{v\}$ to be the graph obtained from G by adding a new vertex v, which is then joined to every vertex in U. In other words, $G \oplus_U \{v\} = (V \cup \{v\}, E \cup \{\{u, v\} : u \in U\})$.

a) Prove that if $G = (V, E)$ is a k-connected simple graph and $U \subseteq V$ has size k, then the graph $G \oplus_U \{v\}$ is k-connected as well.

Solution: Suppose for a contradiction $G' := G \oplus_U \{v\}$ is not k-connected. By Menger's theorem, there exists a vertex cut $S \subseteq V(G')$ of size at most $k - 1$. Clearly, if $v \in S$, then $G' - S$ is actually a subgraph of G with at least $|V| - k - 2$ vertices, which is definitely connected (in fact, it is even 2-connected) by the connectivity assumption on G.

Now consider $v \notin S$. Let C_1 and C_2 be different connected components of $G' - S$. We claim that both C_1 and C_2 contain a vertex from the set V. If not, then one of the components, say C_1, would contain only the vertex v. However, since $|U| = k$, there is at least one vertex $u \in U \setminus S$, and this vertex must be in C_1 as well; a contradiction.

Let $u_1 \in V(C_1) \cap V$ and $u_2 \in V(C_2) \cap V$. It follows that every path in G between u_1 and u_2 have to pass through the set S, which is a contradiction with G being k-connected.

b) For every integer $k > 1$, find a simple graph $G_k = (V_k, E_k)$ on at least $k + 1$ vertices and a vertex-subset $U \subseteq V_k$ of size k such that G_k is not k-connected, however, $G_k \oplus_U \{v\}$ is k-connected.

Solution: *There was a typo in the original statement – one has to assume $k > 1$ since the statement is clearly false for $k = 1$. The points for this part will not be counted to the regular score. You get a bonus point if you have spotted the mistake and constructed a counter-example for the case $k = 1$. You get extra 2 points if you have constructed the graphs G_k for any $k \geq 2$.*

Fix an integer $k \geq 2$. Let $V := \{v_1, v_2, \dots, v_{k+1}\}$ and let $G_k := \left(V, \binom{V}{2} \setminus \{k, k+1\}\right)$. In other words, G_k is obtained from a complete graph on $k+1$ vertices by removing one edge. Clearly, this graph is not k-connected because the set $\{v_1, \dots, v_{k-1}\}$ is a vertex cut in G_k of size $k-1$. Let $U := \{2, 3, \dots, v_{k+1}\}$, and $G'_k := G_k \oplus_U \{v\}$. We claim G'_k is k-connected.

Indeed, consider $S \subseteq V(G'_k)$ a vertex cut in G'_k. By Menger's theorem, it is enough to show $|S| \geq k$. First, observe that for any $i \in \{2, 3, \dots, k-1\}$, the vertex v_i is connected to every other vertex in G'_k. Therefore, any vertex cut in G'_k must contain all the vertices from $\{v_2, v_3, \dots, v_{k-1}\}$, so $|S| \geq k - 2$. But $G'_k - \{v_2, v_3, \dots, v_{k-1}\}$, i.e., the subgraph of G'_k induced by $\{v_1, v_k, v_{k+1}, v\}$, is isomorphic to C_4, so $|S| \geq k - 1$. However, if $|S| = k - 1$, then by the argument above S contains exactly one vertex from $\{v_1, v_k, v_{k+1}, v\}$. Therefore $G'_k - S$ is isomorphic to a path of length two, a contradiction.

2. Let $G = (V, E)$ be a k-connected simple graph and $U, W \subseteq V$ two vertex-subsets, each of size k. Prove that there exist k pairwise vertex-disjoint paths $P_1, \dots P_k$ such that for every $i \in \{1, \dots, k\}$, the path P_i have one endpoint in U and the other endpoint in W.

Solution: Let $G' := (G \oplus_U u) \oplus_W w$. By the part (a) of the previous exercise, G' is k-connected. Therefore, G' contains k internally disjoint paths $Q_1, \dots, Q_k$ between u and w. For every $i \in \{1, \dots, k\}$, let $P_i := Q_i - u - w$. It follows that these are k vertex-disjoint paths in G, each with exactly one end in U and the other in W.

3. Let $G = (V, E)$ be a 2-connected simple graph. Show that for any triple of distinct vertices $u, v, w \in V$ there is a path in G from u to v passing through w, i.e., w is one of the inner vertices of the path.
Solution: Let $G' := G \oplus_U z$ for $U := \{u, v\}$. Again, the first part of Exercise 1 yields that G' is 2-connected. Hence G' contains 2 internally vertex-disjoint paths Q_1 and Q_2 between z and w. Taking their union and removing the vertex z yields the desired path between u and v that passes through w.

4. Let $G = (V, E)$ be a 2-connected simple graph and $v \in V$ a vertex of G. Prove that there exists a vertex $u \in V$ such that $\{u, v\} \in E$ and the graph $G - u - v$ is connected.
Solution: Let U be the set of neighbors of v in G. Let T be a connected subgraph of $G - v$ with the minimum number of edges such that $U \subseteq V(T)$. It is easy to see that T is a tree, and that every leaf of T is a neighbor of v. Let u be a leaf of T. Then $T - u$ is connected. Suppose for a contradiction that $G - u - v$ is not connected and consider a component C of $G - u - v$ which does not contain $T - u$. Thus C contains no neighbor of v and so it is a connected component of $G - u$. It follows that $G - u$ is not connected, contradicting 2-connectivity of G.

5. Let $G = (V, E)$ be a directed graph (digraph) and for each edge $e \in E$, let $\phi(e) \geq 0$ be a non-negative integer. Show that if for every vertex v

$$\sum_{e \in \partial^-(v)} \phi(e) = \sum_{e \in \partial^+(v)} \phi(e) ,$$

then there is a collection of directed cycles $C_1, ..., C_k$ (possibly with repetition) so that for every edge e of G, it holds that

$$|\{i : 1 \leq i \leq k, \ e \in E(C_i)\}| = \phi(e).$$

Solution: Induction on $S := \sum_{e \in E(G)} \phi(e)$. Base case: $S = 0$ is trivial. For the induction step, it suffices to find a directed cycle C in G so that $\phi(e) \geq 1$ for every edge $e \in E(G)$, as one can then apply the induction hypothesis to

$$\phi'(e) := \begin{cases} \phi(e), & \text{if } e \notin E(C) \\ \phi(e) - 1, & \text{if } e \in E(C) \end{cases}$$

Let e be an edge of G with $\phi(e) \geq 1$, a tail u and a head v. Then ϕ restricted to $E(G) - e$ is a v-u-flow of value 1. By Lemma 11.3 from the lecture notes, there exists a directed path P in $G - e$ so that ϕ is positive on every edge of the path. The path P together with e forms the desired cycle.